ECON 6110, Microeconomic Theory III	Section 11
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1 Bayesian Extensive Games and the Perfect Bayesian Equilibrium (PBE)

Definition 1.1. A **Bayesian extensive game with observed actions** is a tuple $\langle N, H, P, (\Theta_i), (p_i), (u_i) \rangle$ where:

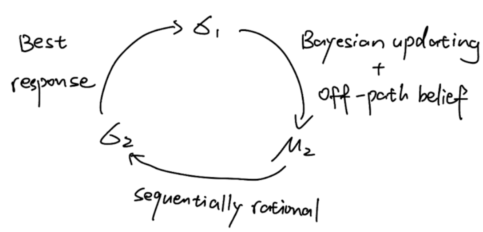
1. Set of N players, set of histories H, and player function P.
2. For each i:
   (a) A finite set of types Θ_i.
   (b) A probability measure p_i over Θ_i. (Assume independent types and common prior)
   (c) A preference relation $\succsim_i$ over $Z \times \Theta$.

Remark. In solving the game, we often recast the game as an extensive game with imperfect information, which is a tuple $\langle N, H, P, f_c, (\mathcal{I}_i), (u_i) \rangle$. We introduce Nature as another player, selecting types at time 0. (It will become clearer in the signaling game)

Definition 1.2 (Informal)**.** An assessment (σ, μ) is a **perfect Bayesian equilibrium** if

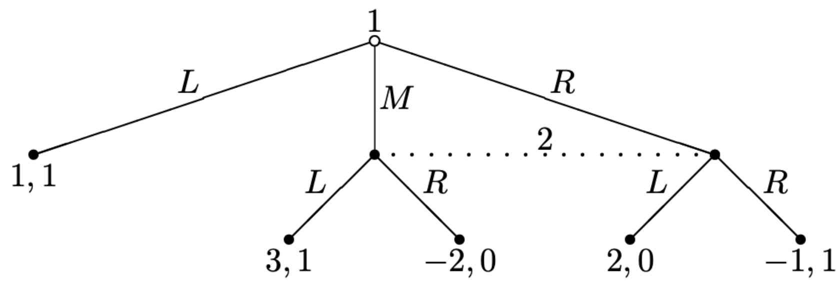
1. Sequentially rational: For each type θ_i, σ_i is the best response given μ_i and σ_{-i} at every information set I_i.
2. Bayesian updating whenever possible. (But no restriction on off-path beliefs)
3. Action determine beliefs: beliefs on i's type can only be changed by i's action. (True when independent types)

Remark. Solving for PBE often proceeds in a "loop":



Remark. This definition applies to general extensive games with imperfect information, not just Bayesian extensive games with observed actions. In the literature, this is sometimes called the "weak sequential equilibrium."

2 Exercises



Find the set of perfect Bayesian equilibria of the game.

Solutions:

Denote player 1's strategy by (α, β, ζ). In all perfect Bayesian equilibria:

• If $\beta > \zeta$ then player 2 chooses L and hence $\beta = 1$; (M,L) is indeed a Perfect Bayesian equilibrium strategy profile.

• If $\beta < \zeta$ then player 2 chooses R, so that player 1 chooses L and $\beta = \zeta = 0$, a contradiction.

• If $\beta = \zeta > 0$ then player 2 must choose L with probability 1/2, in which case player 1

is better off choosing L, a contradiction.

• If $\beta = \zeta = 0$ then player 2's strategy $(\delta, 1 - \delta)$ has to be such that

$$1 \geq 3\delta - 2(1 - \delta) = 5\delta - 2$$

or $3/5 \geq \delta$, and

$$1 \geq 2\delta - (1 - \delta) = 3\delta - 1$$

or $2/3 \geq \delta$. For each $0 < \delta \leq 3/5$ the strategy is supported by the belief $(1/2, 1/2)$ of player 2. For $\delta = 0$ the strategy is supported by any belief $(p, 1 - p)$ with $p \leq 1/2$.

In summary, there are two types of Perfect Bayesian equilibria: one in which the strategy profile is $((0, 1, 0), (1, 0))$ and player 2's belief is $(1, 0)$, and one in which the strategy profile is $((1, 0, 0), (\delta, 1 - \delta))$ for some $\delta \in [0, 3/5]$ and player 2's belief is $(1/2, 1/2)$ for $\delta > 0$ and $(p, 1 - p)$ for some $p \leq 1/2$ for $\delta = 0$.

Remark. An equally valid approach would have been starting out with the strategy of player 2 (σ_2 in the loop), proceed to player 1's best response (σ_1), then derive player 2's belief (μ_2). Finally, we check if σ_2 is sequentially rational given μ_2 and this gives us the PBE.