ECON 6110, Microeconomic Theory III	**Section 10**
TA: Ruqing Xu	**March 28, 2025**

1 Bayesian games and Bayesian Nash equilibrium (BNE)

Definition 1.1.

1. Set of N players.
2. Finite set of state of nature Ω.
3. For each i:
   (a) Action set A_i.
   (b) Set of types T_i and a signal function $\tau_i : \Omega \to T_i$.
   (c) A probability measure p_i over Ω
   (d) A preference relation $\succsim_i$ over $A \times \Omega$

Definition 1.2. A **Bayesian Nash Equilibrium** is the Nash Equilibrium of the strategic game defined as:

- The set of players is the set of pairs (i, t_i) for each $i \in N$ and $t_i \in T_i$.
- The set of actions of player i, t_i is A_i.
- The preference of (i, t_i) is defined as:

$$a^* \succsim b^* \iff L_i(a^*, t_i) \succsim L_i(b^*, t_i)$$

  where $L_i(a, t_i)$ is a lottery over $A \times \Omega$ that assigns probability $\frac{p_i(\omega)}{p_i(\tau_i^{-1}(t_i))}$ to $(a^*_{\tau_j(\omega)}, \omega)$ if $\omega \in \tau_i^{-1}(t_i)$ and 0 otherwise.

Example 1.1.

Remark. Here, the action profile a^* is of length $\sum_i |T_i|$, i.e., it collects actions for each (i, t_i) pair. For example, for a given state ω, $a^*_{\tau_j(\omega)}$ denotes the action of $(j, \tau_j(\omega))$. Therefore, when a player receives a signal t_i, she updates her posterior belief over states, which gives the probability distribution over the opponents' types according to the (deterministic) function τ and the opponents' actions given by the action profile.

Definition 1.3 (Equivalent formulation)**.** In a Bayesian game, each (player, type) pair has a strategy. Equivalently, we can say that each player's strategy is a function that maps from his type to his action.

$$\sigma_i : T_i \to \Delta(A_i)$$

Let $\sigma(t) = (\sigma_1(t_1), \ldots, \sigma_n(t_n))$ denote the strategy profiles. Player i's expected payoff from playing a_i when receiving a signal t_i is:

$$u_i(a_i, \sigma_{-i} \mid t_i) = \sum_{\omega, t_{-i}} u_i(\omega, a_i, \sigma_{-i}(t_{-i}))p(\omega, t_{-i} \mid t_i)$$

So a strategy profile σ is a BNE if and only if for all i, t_i, and a_i such that $\sigma_i(a_i \mid t_i) > 0$:

$$u_i(a_i, \sigma_{-i} \mid t_i) = \max_{a'_i} u_i(a'_i, \sigma_{-i} \mid t_i)$$

Remark. In a Nash equilibrium of a Bayesian game each player chooses the best action available to him given the signal that he receives and his belief about the state and the other players' actions that he deduces from this signal.

2 Exercise: Existence of Pure Strategy BNE in a Simple Game of Incomplete Information

Consider the following static game of incomplete information. There are two states, two players, and two pure actions for each player. In state θ_1, the payoffs for the players are given by

	L	R
U	3,1	0,0
D	4,0	1,3

whereas in state θ_2, the payoffs for the players are given by

	L	R
U	5,0	2,2
D	2,2	0,0

Player 2 receives a signal informing him of the state, whereas Player 1 receives no such signal. Finally, both players agree that the prior probability of state θ_1 is $p \in (0, 1)$ and state θ_2 is $1 - p$.

(a) For what values of p are there no pure strategy BNE?
(b) For what values of p are there multiple pure strategy BNE?
(c) For what values of p is there a unique pure strategy BNE?

3 A Simple Model of Adverse Selection

Firm 1 (the "acquirer") is considering taking over firm 2 (the "target"). Firm 1 does not know firm 2's value; it believes that this value, when firm 2 is controlled by its own management, is uniformly distributed between 0\$ and 100\$. Firm 2 will be worth 50% more under firm 1's management than it is under its own management. Suppose that firm 1 bids $y \geq 0$ to take over firm 2, and firm 2 is worth $x \in [0, 100]$ (under its own management). Then if 2 accepts 1's offer, 1's payoff is $\frac{3}{2}x - y$ and 2's payoff is y; if 2 rejects 1's offer, 1's payoff is 0 and 2's payoff is x.

(a) Model this situation as a *static* game of incomplete information in which *simultaneously* firm 1 chooses how much to offer and firm 2 decides the lowest offer to accept. Specify players, actions, information structure, etc.

(b) Find the Bayes-Nash equilibrium of this game.

(c) Explain why the logic behind the equilibrium is called "adverse selection."