ECON 6110, Microeconomic Theory III	**Section 9**
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1 Repeated Games with Imperfect Public Monitoring

See Fudenberg and Tirole Chapter 5.5 for reference.

Definition 1.1.

1. Let $A_1, ..., A_I$ be finite action sets.

2. Let Y be a finite set of public outcomes.

3. Let $\pi(y \mid a) = \Pr(y \mid a)$. Each action profile induces a probability distribution over the public outcomes.

4. Let $r_i(a_i, y)$ be i's payoff if she plays a_i and the public outcome is y.

5. Player i's expected payoff is:

$$g_i(a) = \sum_{y \in Y} \pi(y \mid a) r_i(a_i, y)$$

6. A mixed strategy is $\alpha_i \in \Delta(A_i)$. Payoffs are defined in the obvious way.

Remark. Player i's realized payoff, $r_i(a_i, y)$, is independent of the actions of the other players (conditional on observing y). Otherwise, player i's payoff could give him private information about his opponents' play.

Definition 1.2.

1. The public information at the start of period t : $h^t = (y^0, ..., y^{t-1})$.

2. Player i's private information is $h_i^t = (a_i^0, ..., a_i^{t-1})$.

3. A public strategy for player i is a sequence of maps $\sigma_i^t : h^t \to \Delta(A_i)$.

Remark. Note that subgame perfection would not be restrictive in these games, since the **only** proper subgame is the game starting from time 0. However, once all players use public strategies there is nothing unobserved that is payoff relevant. Then, the idea of Nash equilibrium in "subgames" can be used again.

Definition 1.3. A profile $(\sigma_1, .., \sigma_I)$ is a perfect public equilibrium if:

1. σ_i is a public strategy for all i.
2. For each date t and history h^t, the strategies are a Nash equilibrium from that point on.

Example 1.1 (Oligopoly and trigger-price strategies). Cournot competition with noisy demand (Green and Porter, 1984). Firms set outputs $q_1^t, ...q_I^t$, chosen privately. Demand conditions then determine $p^t = P(q_1^t, ...q_I^t, \epsilon)$, which is observed publicly. Consider the strategy:

1. Play $q_1, ...q_I$. If $p^t < \underline{p}$, go to phase 2.
2. Play $q_1^c, ..., q_I^c$ (Cournot) for T periods. Then return to phase I.

In lecture we showed that this strategy can sustain $q_1, ...q_I$ smaller than the Cournot outcome.

Remark. Note that in equilibrium, all players correctly forecast that their opponents will never deviate. Thus, all players know that a low price is not triggered by some players' deviation but by noise. However, they carry out the "punishment" anyway according to the equilibrium strategy. The existence of punishment serves to make player's future action contingent on current actions, and thus provide incentive for players to cooperate.

Example 1.2 (Noisy prisoner's dilemma). Consider prisoner's dilemma and let the set of outcomes Y equal the set of action profiles A, but y doesn't have to correspond to a. For example, if both players played $a_i = C$, the distribution over public outcomes could be:

$$\pi((c, c) \mid (C, C)) = (1 - \epsilon)^2$$
$$\pi((c, d) \mid (C, C)) = \epsilon(1 - \epsilon)$$
$$\pi((d, c) \mid (C, C)) = \epsilon(1 - \epsilon)$$
$$\pi((d, d) \mid (C, C)) = \epsilon^2$$

For some ϵ small. The key assumption here is that the "intended actions" (C,D) are not observed, only the realized ones (c,d), and only the realized actions affect players' payoffs. Recall:

Remark. Player i's realized payoff, $r_i(a_i, y)$, is independent of the actions of the other players (conditional on observing y). Otherwise, player i's payoff could give him private information about his opponents' play.

So it's better to think it as a situation where each player has probability ϵ of making a "mistake."

Definition 1.4. The pair (α, v) is **enforceable** with respect to δ and $W \subset \mathbb{R}^I$ if there exists a function $w : Y \to W$ such that for all i,

1. (correctly specified)

$$v_i = (1-\delta)g_i(\alpha) + \delta \sum_y \pi(y \mid \alpha) w_i(y)$$

2. (optimal)

$$\alpha_i \in \arg\max_{\alpha_i' \in \Delta(A_i)} (1-\delta)g_i(\alpha_i', \alpha_{-i}) + \delta \sum_y \pi(y \mid \alpha_i', \alpha_{-i}) w_i(y)$$

Remark. $w_i(y)$ is the future expected value normalized as average per period value. So the entire future expected value would be $\frac{w_i(y)}{1-\delta}$, and thus cancelled by $(1-\delta)$.

Definition 1.5. If for some α, (α, v) is enforceable with respect to δ and W, we say that v is **generated** by (δ, W). The set of all payoffs v generated by (δ, W) is denoted $B(\delta, W)$.

Remark. To prove that $v \in B(\delta, W)$ (v can be generated by (δ, W)), all we need is to find a stage game action profile α and a map $w : Y \to W$ such that the two conditions in Definition 1.4 are satisfied.

Definition 1.6. W is self-generating if $W \subset B(\delta, W)$. In other words, the payoff vectors in W can be enforced with continuation payoffs in W.

Remark. To prove that W is self-generating, all we need is to prove the previous remark for every $v \in W$.

Theorem 1.3. *$E(\delta)$ is the set of PPE payoffs. If W is self-generating, then $W \subset E(\delta)$: All payoffs in W are PPE payoffs.*

2 Exercises

Consider the prisoner's dilemma with perfect monitoring.

	C	D
C	$1, 1$	$-1, 2$
D	$2, -1$	$0, 0$

With perfect monitoring, $Y = \{(C, C), (C, D), (D, C), (D, D)\}$.

Exercise 2.1. Let $W = \{(0, 0), (1, 1)\}$. Try showing that if $\delta = 1/2$, then $(3/2, 0)$ is in $B(\delta, W)$.

Answers: Let $\alpha = (D, C)$. Let

$$w(C, C) = (1, 1)$$
$$w(C, D) = (0, 0)$$
$$w(D, C) = (1, 1)$$
$$w(D, D) = (0, 0)$$

Check condition (1):

$$(3/2, 0) = (1 - \delta)g(D, C) + \delta w(D, C) = 1/2(2, -1) + 1/2(1, 1)$$

Check condition (2):

$$3/2 \geq (1 - \delta)g_1(C, C) + \delta w_1(C, C) = 1/2(1) + 1/2(1)$$

$$0 \geq (1 - \delta)g_2(D, D) + \delta w_2(D, D) = 1/2(0) + 1/2(0)$$

Exercise 2.2. Show that if $\delta \geq 1/2$, then the set $W\{(0, 0), (1, 1)\}$ is self-generating.

Answers: Now you should be able to read the proof in lecture slides!

3 Repeated Games with Imperfect Private Monitoring

Example 3.1 (Two-period game). Period 1, prisoner's dilemma:

	C	D
C	1,1	-1,2
D	2,-1	0,0

Period 2, a coordination game ($k > 2$):

	G	B
G	k,k	0,0
B	0,0	1,1

Private, independent monitoring.

$$\Pr(y_i = c \mid a_j) = \begin{cases} 1-\epsilon & a_j = C \\ \epsilon & a_j = D \end{cases}$$

Note that i's signal contains no information about j's signal. In this case, we show that no *pure-strategy equilibrium* support (C, C) in the first period.

Consider as before, a strategy for both players that cooperates C in the first period and coordinates on G iff they observe the good signal c after the first period. Otherwise, play B to punish.

The problem is that i would rationally ignore her signal in the second period and just condition her second-period action on her first-period action. This is because so long as i cooperates in the first period, she will assign high probability to j observing a good signal. Consequently, she prefers to just keep cooperating (not following the strategy to punish). In short, the private monitoring means there is no way to coordinate on the punishment equilibrium in period two following a bad outcome.

However, later in lecture we showed that in this setting, the players may be able to correlate their beliefs by playing mixed strategies in the first period. Consider the following strategies.

- Period 1 : Play C and D with probabilities $\alpha, 1-\alpha$.
- Period 2 : Play G if and only if $a_i = C$ and $y_i = c$.

This strategy sustains an equilibrium. Note that a key to the randomization equilibrium is that player i conditions his second period behavior on the result of his first period randomization. Because of this, player j's signal (generated from i's first-period action) is informative about i's second period behavior. This means that player j will want to condition his second period action on his first period signal, which means in turn that some incentive can be provided for i to cooperate in the first period.

Private, correlated monitoring.

$$\Pr(y_1 = c, y_2 = c \mid (a_1, a_2)) = \begin{cases} 1 - \epsilon & (a_1, a_2) = (C, C) \\ \epsilon & else \end{cases}$$

Strategies below is an equilibrium.

- Period 1 : Play C.
- Period 2 : Play G if and only if $y_i = c$.

Remark. Here the game is in fact *public monitoring* (a form of noisy prisoner's dilemma). But in general, if y_i and y_j are highly, but not perfectly, correlated, it will be possible to support cooperation in the first period by coordinating on different second period play depending on the signals.