ECON 6110, Microeconomic Theory III	Section 7
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1 Repeated game

Definition 1.1. A **repeated game** consists of:

(i) Set of players: $N = \{1, \ldots, n\}$

(ii) Set of actions for each player, A_i

(iii) $T+1$ stages: the game can be finite horizon ($T < \infty$) or infinite horizon ($T = \infty$)

(iv) Payoff function in the stage game for each player, $g_i : A \to \mathbb{R}$

(v) Discount factor, $\delta \in (0, 1]$ ($(0, 1)$ if infinite horizon)

(vi) Period-t history $h^t = (a_1^0, \ldots, a_I^0), \ldots, (a_1^{t-1}, \ldots, a_I^{t-1})$

(vii) Set of period-t histories, $H^t = \{a^0, \ldots, a^{t-1}\}$. Terminal histories Z

(viii) Period-t strategy $s_i^t : H^t \to \Delta(A_i)$. Strategy $s_i = (s_i^t)_{t=0}^T$

(ix) Payoff function for finitely repeated games, $u_i : Z \to \mathbb{R}$,

$$u_i(s_i, s_{-i}) = \frac{1-\delta}{1-\delta^{T+1}} \sum_{t=0}^{T} \delta^t g_i(s_i(h^t), s_{-i}(h^t))$$

(x) Payoff function for infinitely repeated games, $u_i : Z \to \mathbb{R}$,

$$u_i(s_i, s_{-i}) = (1-\delta) \sum_{t=0}^{\infty} \delta^t g_i(s_i(h^t), s_{-i}(h^t))$$

Theorem 1.1 (One-stage deviation principle)**.** *In an infinitely-repeated game, a strategy profile s is an SPE if and only if for all players i, all histories $h \in H$, and one-stage deviations $\hat{s}_i$,*

$$u_i(s_i |_h, s_{-i} |_h) \geq u_i(\hat{s}_i, s_{-i} |_h)$$

Proposition 1.2. *Let α be a Nash equilibrium of the stage game. Consider the strategy profile s such that for all i and all h^t,*

$$s_i(h^t) = \alpha_i$$

Then s is an SPE. That is, any Nash equilibrium of the stage game, when repeated in each period, is an SPE of the repeated game.

Example 1.3. Consider the finitely-repeated prisoner's dilemma game:

$$\begin{array}{c|cc} & C & D \\ \hline C & 2,2 & 0,3 \\ D & 3,0 & 1,1 \end{array} \rightarrow \begin{array}{c|cc} & C & D \\ \hline C & 2,2 & 0,3 \\ D & 3,0 & 1,1 \end{array} \rightarrow \cdots \rightarrow \begin{array}{c|cc} & C & D \\ \hline C & 2,2 & 0,3 \\ D & 3,0 & 1,1 \end{array}$$

The unique stage game Nash equilibrium is (D, D). Consider the candidate repeated game equilibrium given by $(\text{always play } D, \text{always play } D)$. Assume $\delta = 1$. We can see that this is an SPE by looking at one-shot deviations. The payoff to not deviating at a history $h^t = (a^0, \ldots, a^{t-1})$ is

$$\frac{1}{T}\left(\sum_{s=0}^{t-1} g_i(a^s) + \sum_{s=t}^{T} 1\right)$$

which exceeds the payoff to deviating to C at h^t:

$$\frac{1}{T}\left(\sum_{s=0}^{t-1} g_i(a^s) + 0 + \sum_{s=t+1}^{T} 1\right)$$

Proposition 1.4. *Suppose $T < \infty$. If the stage game has a unique Nash equilibrium, then the repeated game has a unique SPE, namely, the repetition of that Nash equilibrium in each stage game.*

Proof. Both existence and uniqueness are proven by backward induction. □

Example 1.5. Consider again the finitely-repeated prisoner's dilemma game, of Example 1.3. In the final stage game, to obtain an SPE, both players must play D. Given this, in the penultimate stage game, both players must also play D. By backward induction, we obtain the unique SPE: (always play D, always play D).

Proposition 1.6 (Grim-trigger strategy)**.** *In the infinitely-repeated prisoner's dilemma, for sufficiently large δ, the following strategy defines a symmetric SPE:*

$$s_i(h^t) = \begin{cases} C & \text{if } t = 0 \text{ or } h^t = ((C, C), \ldots, (C, C)) \\ D & \text{otherwise} \end{cases}$$

In words, both players cooperate unless and until one defects, whereafter both players defect forever.

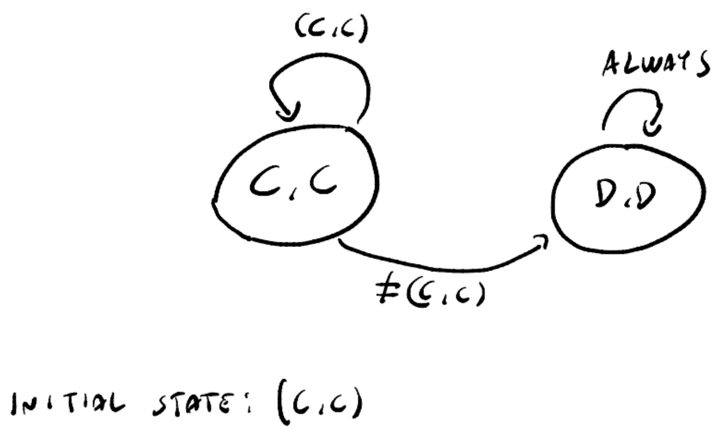

Figure 1: Phase diagram of grim-trigger strategy. There are two states, one in which both player intend to cooperate, and one in which both players intend to defect. The latter is an absorbing state: once in the state, players cannot leave it by any deviation.

Proof. There are two classes of histories: (i) histories where (C, C) is supposed to be played; and (ii) histories where (D, D) is supposed to be played. We must consider possible one-stage deviations for the two classes: (i) playing D initially or after some history of consistent co-operation; and (ii) playing C after someone has deviated. The latter is trivially unprofitable (already playing Nash, no impact on future play), yielding a loss of δ^{t+1} if the deviation occurs in stage t. To see that (i) is unprofitable for sufficiently large δ, let

$$(1-\delta)\left(\sum_{s=0}^{t-1} g_i(a^s) + 2\sum_{s=t}^{\infty}\delta^s\right) \geq (1-\delta)\left(\sum_{s=0}^{t-1} g_i(a^s) + 3\delta^t + \sum_{s=t+1}^{\infty}\delta^s\right)$$

$$2\sum_{s=0}^{\infty}\delta^s \geq 3 \ + \delta\sum_{s=0}^{\infty}\delta^s \qquad \text{(renormalize time at 0)}$$

$$\frac{2}{1-\delta} \geq 3 + \frac{\delta}{1-\delta}$$

$$\frac{2-\delta}{1-\delta} \geq 3$$

$$\delta \geq \frac{1}{2}$$

□

Proposition 1.7 (Limited punishment)**.** *In the infinitely-repeated prisoner's dilemma, for $\delta > \frac{1}{2}$, the following strategy is a SPE:*

$$s_i(h^t) = \begin{cases} C & \text{if } t = 0 \text{ or } (C,C) \text{ was played in the previous stage game} \\ & \text{or } (D,D) \text{ was played in the previous } T \text{ stage games} \\ D & \text{otherwise} \end{cases}$$

That is, any deviation from cooperation is punished with a T-period sequence of defections, which is reset each time the opposing player deviates from D.

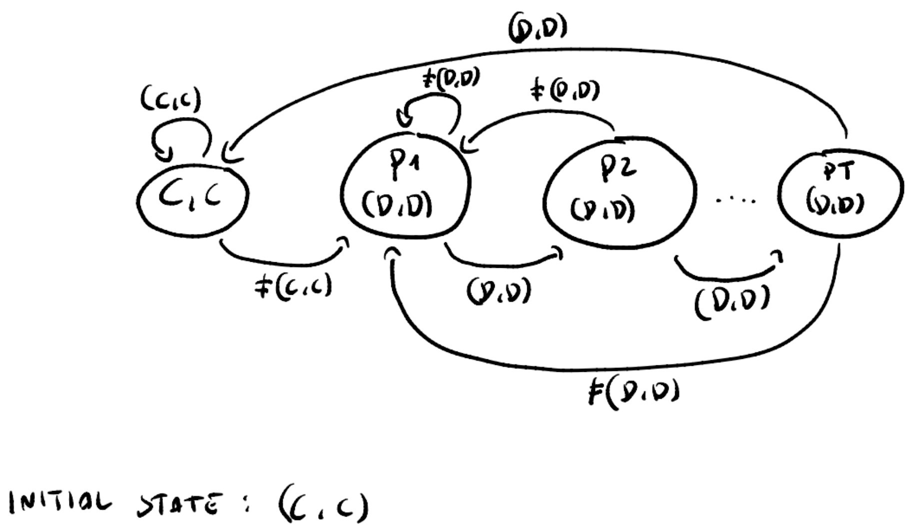

Figure 2: Phase diagram of limited punishment.

Proof. This also defines an SPE for sufficiently large δ, perhaps a more reasonable one than the excessively punitive grim-trigger strategy. To see this, first consider the (C, C) state and the payoff to deviating from this state to D. We must have

$$\begin{aligned}
\frac{2}{1-\delta} &\geq 3 + \delta + \cdots + \delta^T + \delta^{T+1}\frac{2}{1-\delta} \\
\frac{2}{1-\delta} &\geq 3 + \frac{\delta(1-\delta^T)}{1-\delta} + \delta^{T+1}\frac{2}{1-\delta} \\
\delta^{T+1} &\leq 2\delta - 1 \\
(T+1)\log\delta &\leq \log(2\delta - 1) \\
T &\geq \frac{\log(2\delta-1)}{\log\delta} - 1
\end{aligned}$$

If $\delta \leq \frac{1}{2}$, this will not be defined. For $\delta > \frac{1}{2}$, the closer δ is to $\frac{1}{2}$, the higher T must be to sustain cooperation – the more impatient the players, the more severe the punishment must be to ensure cooperation. Now let's consider the other problem, of ensuring that players do indeed play (D, D) in response to some deviation from the on-path strategy profile. It suffices to consider only the state where players intend to play (D, D) for the next J stages, as deviating later in the punishment process is strictly more costly (so if the players won't do so at the first punishment stage, they won't do so subsequently). But deviating at this first punishment stage simply incurs a loss of δ^t in the stage t game – it is unprofitable, so we have an SPE. □

Proposition 1.8 (Tit-for-tat strategy)**.** *In the infinitely-repeated prisoner's dilemma, for* $\delta = \frac{1}{2}$*, the following strategy defines a symmetric SPE:*

$$s_i(h^t) = \begin{cases} C & \textit{if } t = 0 \textit{ or } a_{-i}^{t-1} = C \\ D & \textit{otherwise} \end{cases}$$

In words, both players initially cooperate and mimic the other's strategy in the last period.

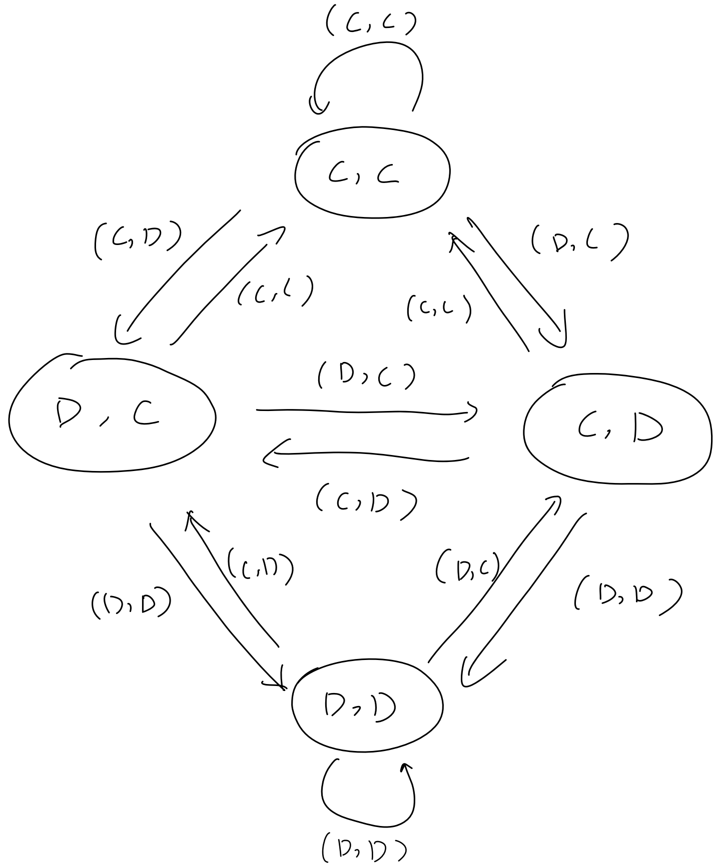

Figure 3: Phase diagram of Tit-for-tat.

Proof. There are four states – two of which are symmetric. We need to show that none of the one-stage deviations in those states are profitable: (i) In state (C, C), one of the players plays D.

$$
\begin{aligned}
2 + 2\delta + 2\delta^2 + 2\delta^3 + 2\delta^4 + \cdots &\geq 3 + 0 + 3\delta^2 + 0 + 3\delta^4 + \cdots \\
\frac{2}{1-\delta} &\geq \frac{3}{1-\delta^2} \\
2(1+\delta) &\geq 3 \\
\delta &\geq \frac{1}{2}
\end{aligned}
$$

(ii) In state (D, C), the player supposed to play C deviates to D.

$$
\begin{aligned}
0 + \delta\frac{3}{1-\delta^2} &\geq \frac{1}{1-\delta} \\
\delta &\geq \frac{1}{2}
\end{aligned}
$$

(iii) In state (D, C), the player supposed to play D deviates to C.

$$\frac{3}{1-\delta^2} \geq \frac{2}{1-\delta}$$
$$\delta \leq \frac{1}{2}$$

(iv) In state (D, D), one of the players plays C.

$$\frac{1}{1-\delta} \geq 0 + \delta\frac{3}{1-\delta^2}$$
$$\delta \leq \frac{1}{2}$$

Therefore, for this to be an SPE, we need that $\delta = \frac{1}{2}$. □

2 Microeconomic Theory Prelim Exam, 2022: Question II

Consider an infinitely repeated game where the stage game is the game of chicken:

	Swerve	*Straight*
Swerve	$0, 0$	$-1, 1$
Straight	$1, -1$	$-10, -10$

Assume that the discount factor is very close to one. For concreteness, you can assume that $\delta = 0.99$.

(a) Is there an SPE in which player 1's payoff is $1/(1-\delta)$? Explain.

(b) Is there an SPE in which each player gets a payoff of 0? Explain.

(c) Is there an SPE in which player 1's payoff is $-2/(1-\delta)$? Explain.

Solution:

(a) Yes: It is enough to repeat the equilibrium of the stage game (Straight, Swerve).

(b) Yes: Consider the following strategy profile: Play (Swerve, Swerve) and keep playing (Swerve, Swerve) unless some player deviates. If player 1 deviates, play (Swerve, Straight) forever. If player 2 deviates, play (Straight, Swerve) forever (they are stage NEs).

You can verify that limited punishment strategy where the players play (Straight, Straight) for 1 period following any deviation is also a SPE.

(c) No: player 1 can be sure of obtaining at least $-1/(1-\delta)$ by always swerving regardless of what the opponent does. In fact, -1 is player 1's minimax payoff, so any Nash equilibrium cannot sustain a payoff lower than the minimax.