ECON 6110, Microeconomic Theory III	**Section 7**
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1 Repeated game

Definition 1.1. A **repeated game** consists of:

(i) Set of players: $N = \{1, \dots, n\}$

(ii) Set of actions for each player, A_i

(iii) $T + 1$ stages: the game can be finite horizon ($T < \infty$) or infinite horizon ($T = \infty$)

(iv) Payoff function in the stage game for each player, $g_i : A \to \mathbb{R}$

(v) Discount factor, $\delta \in (0, 1]$ ($(0, 1)$ if infinite horizon)

(vi) Period-t history $h^t = (a_1^0, \dots, a_I^0), \dots, (a_1^{t-1}, \dots, a_I^{t-1})$

(vii) Set of period-t histories, $H^t = \{a^0, \dots, a^{t-1}\}$. Terminal histories Z

(viii) Period-t strategy $s_i^t : H^t \to \Delta(A_i)$. Strategy $s_i = (s_i^t)_{t=0}^T$

(ix) Payoff function for finitely repeated games, $u_i : Z \to \mathbb{R}$,

$$u_i(s_i, s_{-i}) = \frac{1-\delta}{1-\delta^{T+1}} \sum_{t=0}^{T} \delta^t g_i(s_i(h^t), s_{-i}(h^t))$$

(x) Payoff function for infinitely repeated games, $u_i : Z \to \mathbb{R}$,

$$u_i(s_i, s_{-i}) = (1-\delta) \sum_{t=0}^{\infty} \delta^t g_i(s_i(h^t), s_{-i}(h^t))$$

Theorem 1.1 (One-stage deviation principle)**.** *In an infinitely-repeated game, a strategy profile s is an SPE if and only if for all players i, all histories $h \in H$, and one-stage deviations $\hat{s}_i$,*

$$u_i(s_i\,|_h, s_{-i}\,|_h) \geq u_i(\hat{s}_i, s_{-i}\,|_h)$$

Proposition 1.2. *Let α be a Nash equilibrium of the stage game. Consider the strategy profile s such that for all i and all h^t,*

$$s_i(h^t) = \alpha_i$$

Then s is an SPE. That is, any Nash equilibrium of the stage game, when repeated in each period, is an SPE of the repeated game.

Example 1.3. Consider the finitely-repeated prisoner's dilemma game:

$$\begin{array}{c|cc} & C & D \\ \hline C & 2,2 & 0,3 \\ D & 3,0 & 1,1 \end{array} \rightarrow \begin{array}{c|cc} & C & D \\ \hline C & 2,2 & 0,3 \\ D & 3,0 & 1,1 \end{array} \rightarrow \cdots \rightarrow \begin{array}{c|cc} & C & D \\ \hline C & 2,2 & 0,3 \\ D & 3,0 & 1,1 \end{array}$$

The unique stage game Nash equilibrium is (D, D). Consider the candidate repeated game equilibrium given by $(\text{always play } D, \text{always play } D)$. Assume $\delta = 1$. We can see that this is an SPE by looking at one-shot deviations. The payoff to not deviating at a history $h^t = (a^0, \ldots, a^{t-1})$ is

$$\frac{1}{T}\left(\sum_{s=0}^{t-1} g_i(a^s) + \sum_{s=t}^{T} 1\right)$$

which exceeds the payoff to deviating to C at h^t:

$$\frac{1}{T}\left(\sum_{s=0}^{t-1} g_i(a^s) + 0 + \sum_{s=t+1}^{T} 1\right)$$

Proposition 1.4. *Suppose $T < \infty$. If the stage game has a unique Nash equilibrium, then the repeated game has a unique SPE, namely, the repeated of that Nash equilibrium in each stage game.*

Proof. Both existence and uniqueness are proven by backward induction. □

Example 1.5. Consider again the finitely-repeated prisoner's dilemma game, of Example 1.3. In the final stage game, to obtain an SPE, both players must play D. Given this, in the penultimate stage game, both players must also play D. By backward induction, we obtain the unique SPE: (always play D, always play D).

Proposition 1.6 (Grim-trigger strategy)**.** *In the infinitely-repeated prisoner's dilemma, for sufficiently large δ, the following strategy defines a symmetric SPE:*

$$s_i(h^t) = \begin{cases} C & \text{if } t = 0 \text{ or } h^t = ((C,C), \ldots, (C,C)) \\ D & \text{otherwise} \end{cases}$$

In words, both players cooperate unless and until one deviates, whereafter both players deviate forever.

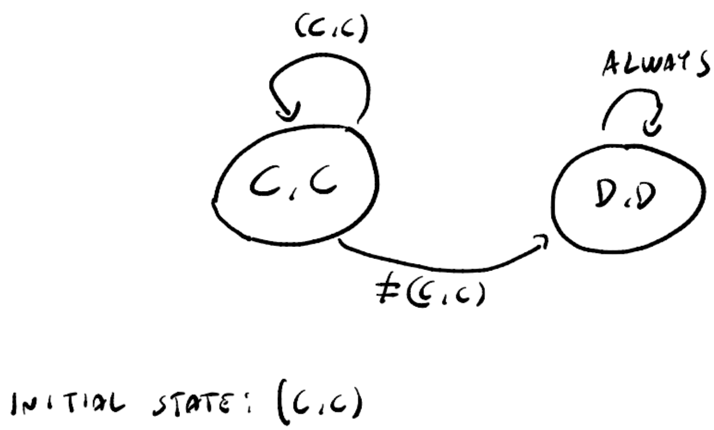

Figure 1: Phase diagram of grim-trigger strategy.

Proposition 1.7 (Limited punishment)**.** *In the infinitely-repeated prisoner's dilemma, for $\delta > \frac{1}{2}$, the following strategy is a SPE:*

$$s_i(h^t) = \begin{cases} C & \textit{if } t = 0 \textit{ or } (C, C) \textit{ was played in the previous stage game} \\ & \textit{or } (D, D) \textit{ was played in the previous } T \textit{ stage games} \\ D & \textit{otherwise} \end{cases}$$

That is, any deviation from cooperation is punished with a T-period sequence of defections, which is reset each time the opposing player deviates from D.

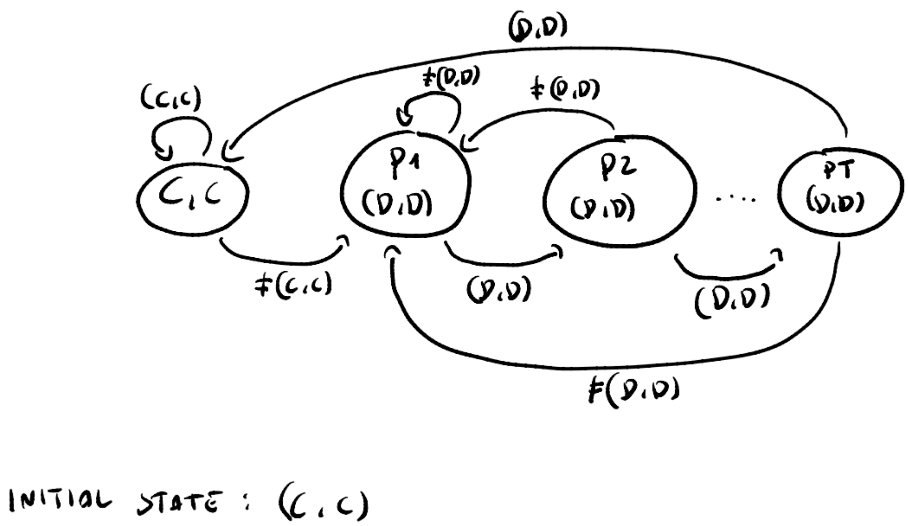

Figure 2: Phase diagram of limited punishment.

Proposition 1.8 (Tit-for-tat strategy)**.** *In the infinitely-repeated prisoner's dilemma, for $\delta = \frac{1}{2}$, the following strategy defines a symmetric SPE:*

$$s_i(h^t) = \begin{cases} C & \textit{if } t = 0 \textit{ or } a_{-i}^{t-1} = C \\ D & \textit{otherwise} \end{cases}$$

In words, both players initially cooperate and mimic the other's strategy in the last period.

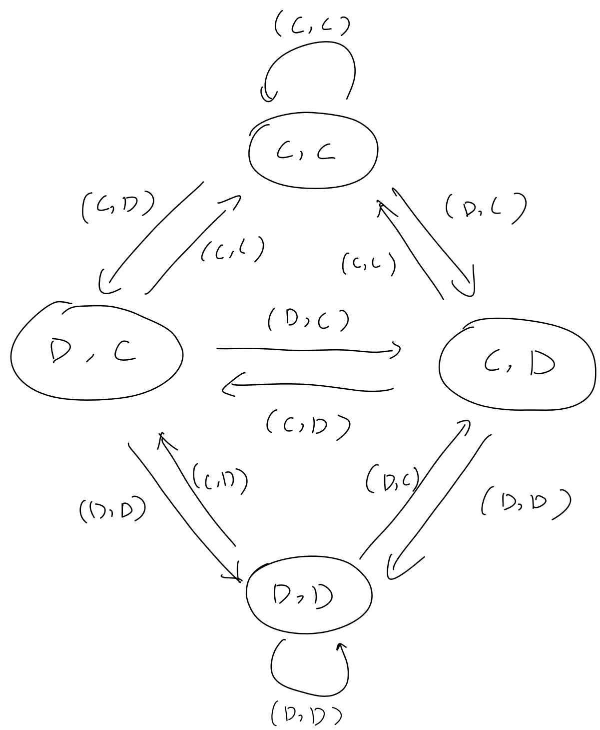

Figure 3: Phase diagram of Tit-for-tat.

2 Microeconomic Theory Prelim Exam, 2022: Question II

Consider an infinitely repeated game where the stage game is the game of chicken:

	Swerve	*Straight*
Swerve	$0, 0$	$-1, 1$
Straight	$1, -1$	$-10, -10$

Assume that the discount factor is very close to one. For concreteness, you can assume that $\delta = 0.99$.

(a) Is there an SPE in which player 1's payoff is $1/(1-\delta)$? Explain.

(b) Is there an SPE in which each player gets a payoff of 0? Explain.

(c) Is there an SPE in which player 1's payoff is $-2/(1-\delta)$? Explain.