ECON 6110, Microeconomic Theory III	**Section 6**
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1 Bargaining and the existence

Definition 1.1. Rubinstein's bargaining model consists of:

(i) Two players: $N = \{1, 2\}$

(ii) They bargain over the division of a pie of size one $(\alpha, 1-\alpha)$, $\alpha \in [0, 1]$

(iii) Individual discount factors, $\delta_i \in (0, 1)$

(iv) In even periods $t = 0, 2, \ldots$

- (a) Player 1 offers $(\alpha_1, 1-\alpha_1)$
- (b) Player 2 accepts or rejects
- (c) If player 2 accepts, game ends
- (d) Payoffs: $\delta_1^t \alpha_1$ and $\delta_2^t (1-\alpha_1)$
- (e) If player 2 rejects, game proceeds to the next period

(v) In odd periods $t = 1, 3, \ldots$

- (a) Player 2 offers $(\alpha_2, 1-\alpha_2)$
- (b) Player 1 accepts or rejects
- (c) If player 1 accepts, game ends
- (d) Payoffs: $\delta_1^t (1-\alpha_2)$ and $\delta_2^t \alpha_2$
- (e) If player 1 rejects, game proceeds to the next period

Definition 1.2. A strategy of player i is **stationary** if for any history after which it is player i's turn to propose an agreement she proposes the same agreement, and for any history after which it is her turn to respond to a proposal she uses the same criterion to choose her response.

Question 1: Assume stationary strategies. Find a subgame perfect equilibrium.

Proof. Consider the stationary strategies where when i is the initiator, i offers $(\alpha_i, 1-\alpha_i)$, where α_i is the fraction they keep for themselves. The respondent accepts any offer that is weakly more generous than $1-\alpha_i$. To make sure this strategy is an SPE, we need to check all histories and all one-stage deviations. Recall that here all histories means all records of past actions.

A better thing to do is to partition the possible histories into four cases, and look at the possible one shot deviations in each of these cases:

(i) It is your turn to offer. Your opponent accepting your offer must be at least as profitable as waiting and accepting their counteroffer: $\alpha_i \geq \delta_i(1-\alpha_{-i})$.

(ii) Your opponent has just offered $1-\alpha_{-i}$. Accepting your opponent's offer must be at least as profitable as waiting and having your counteroffer accepted in the following period: $1-\alpha_{-i} \geq \delta_i\alpha_i$.

(iii) Your opponent has just offered $1-\beta_{-i}$, $\beta_{-i} < \alpha_{-i}$. Accepting your opponents offer must be at least as profitable as waiting and having your counteroffer accepted in the following period: $1-\beta_{-i} \geq \delta_i\alpha_i$. This condition is implied by the previous one.

(iv) Your opponent has just offered $1-\beta_{-i}$, $\beta_{-i} > \alpha_{-i}$. It must be at least as profitable for you to wait one period and have your counteroffer accepted: $\delta_i\alpha_i \geq 1-\beta_{-i}$ for all $\beta_{-i} > \alpha_{-i}$. Letting $\beta_{-i} = \alpha_{-i} + \epsilon$ and $\epsilon \to 0$, we have $\delta_i\alpha_i \geq 1-\alpha_{-i}$.

Together, (ii) and (iv) imply $1-\alpha_{-i} = \delta_i\alpha_i$. This implies

$$\begin{aligned}
\delta_j\delta_i\alpha_i &= \delta_j - \delta_j\alpha_j = \delta_j - (1-\alpha_i) = \delta_j - 1 + \alpha_i \\
\implies \alpha_i &= \frac{1-\delta_j}{1-\delta_j\delta_i} \\
\implies 1-\alpha_i &= \frac{\delta_j - \delta_j\delta_i}{1-\delta_j\delta_i}
\end{aligned}$$

Check that condition (i) is satisfied:

$$1-\delta_j > \delta_i^2(1-\delta_j) = \delta_i(\delta_i - \delta_i\delta_j)$$

By the one-shot deviation principle, this is a subgame perfect equilibrium. □

Remark (Properties of subgame perfect equilibrium).

(i) Immediate agreement: In this subgame perfect equilibria, agreement is reached immediately. This is Pareto efficient.

(ii) First mover advantage:

$$\delta_1 = \delta_2 = \delta \qquad \Longrightarrow \qquad \alpha_1 = \frac{1-\delta}{1-\delta^2} > \frac{1}{2}$$

(iii) Impatience matters: the more impatient a player the worse off she is in equilibrium

$$\frac{\partial}{\partial \delta_1}\left(\frac{1-\delta_2}{1-\delta_1\delta_2}\right) > 0 \quad \text{and} \quad \frac{\partial}{\partial \delta_2}\left(\frac{1-\delta_2}{1-\delta_1\delta_2}\right) < 0$$

Proposition: Consider a two player, infinite-horizon bargaining game and a stationary strategy profile wherein Player i offers $(\alpha_i, 1-\alpha_i)$ and accepts offers by the other player according the payoff threshold $1-\alpha_{-i}$. For any partial history in which i is the acceptor, let z_i denote her expected payoff were play to continue into the next round. Additionally, for any partial history in which i is the offerer, let w_i denote her expected payoff were play to continue into the next round. Then, the stated strategy profile is a SPNE if and only if $z_i = 1-\alpha_{-i}$ and $\alpha_i \geq w_i$.

Proof:($\Longrightarrow$) Suppose that $z_i > 1-\alpha_{-i}$. Then, in any partial history wherein Player $-i$ offers Player i an amount $y \in (z_i, 1-\alpha_{-i})$, i would have a profitable one shot deviation to reject the offer. Conversely, suppose that $z_i < 1-\alpha_{-i}$. Then, in any partial history in which $-i$ offers $y \in (z_i, 1-\alpha_{-i})$, i would have a profitable one shot deviation to accept the offer. Thus, $z_i = 1-\alpha_{-i}$. Finally, suppose that $w_i > \alpha_i$. Then, in any partial history in which i is the offerer, she would have a profitable one shot deviation to propose $(y, 1-y)$ with $y > \alpha_i$. Thus, $\alpha_i \geq w_i$.

($\Longleftarrow$) First, consider a partial history in which i the offerer. Suppose that she has a profitable one shot deviation to offer $(y, 1-y)$ with $y < \alpha_i$. Then, $y > \alpha_i$, contradiction. Conversely, suppose that she has a profitable one shot deviation to offer $(y, 1-y)$ with $y > \alpha_i$. Then, since $-i$ will reject the offer, it must the case that $w_i > \alpha_i$, a contradiction. Now, consider a partial history wherein player i has a profitable one shot deviation to accept some offer $y < 1-\alpha_{-i}$. Then, we must have that $y > z_i$, which implies that $1-\alpha_{-i} > z_i$, a contradiction. Conversely, consider a partial history wherein player i has a profitable one shot deviation to reject some offer $y \geq 1 - \alpha_{-i}$. Then, we must have that $z_i > y$, which implies that $z_i > 1 - \alpha_{-i}$, a contradiction. Since we have found a contradiction for all conceivable one shot deviations, we conclude that the stated strategy profile is a SPNE.

2 Exercises

Consider the alternating-offer bargaining game that we have discussed in class, with the following modification: At the beginning of every period, a coin is toss. If it comes up Heads, player 1 is the initiator who makes the offer. If it comes up Tails, player 2 is the initiator who makes the offer. For this exercise, you may assume that the players have the same discount factor $\delta \in (0, 1)$.

(a) Assume first that the two players agree that the probability of Heads is $p \in (0, 1)$. Find a subgame perfect equilibrium.

(b) Assume now that the two players disagree on the probability of Heads: player 1 believes that the probability of Heads is $p_1 \in (0, 1)$, while player 2 believes that the probability of Tails is $p_2 \in (0, 1)$. The disagreement is commonly known: they agree to disagree. Find a subgame perfect equilibrium.

Proof. Although the players have the same discount factor, they still have different bargaining power due to their different probability of being the initiator. Suppose that $(\alpha_i, 1 - \alpha_i)$ is the offering and acceptance threshold for player 1 and 2. We check all possible one-shot deviations and find conditions for making them not profitable.

(i) At all histories at which 1 is the initiator. The original strategy is to offer α_1 and get α_1. Possible one-shot deviations are: 1) offering $\beta_1 \leq \alpha_1$ and get β_1, and 2) offering $\beta_1 \geq \alpha_1$ and be rejected. In the second stage, with probability p player 1 makes the offer again and gets $\delta\alpha_1$, and with probability $1 - p$ player 2 makes the offer instead and player 1 gets $\delta(1 - \alpha_2)$. The condition for non-profitable one-shot deviations is:

$$\alpha_1 \geq p\delta\alpha_1 + (1 - p)\delta(1 - \alpha_2)$$

(ii) At all histories at which 2 is the initiator. Following the same logic, we can derive the condition for non-profitable one-shot deviations:

$$\alpha_2 \geq p\delta(1 - \alpha_1) + (1 - p)\delta\alpha_2$$

(iii) At all histories at which 1 is the recipient and is offered $1 - \alpha_2$. The original strategy is to accept and get $1 - \alpha_2$. Possible one-shot deviations is: 1) reject and get $p\delta\alpha_1 +$

$(1-p)\delta(1-\alpha_2)$. the condition for non-profitable one-shot deviations is:

$$1-\alpha_2 \geq p\delta\alpha_1 + (1-p)\delta(1-\alpha_2)$$

(iv) At all histories at which 2 is the recipient and is offered $1-\alpha_1$. Following the same logic, we can derive the condition for non-profitable one-shot deviations:

$$1-\alpha_1 \geq p\delta(1-\alpha_1) + (1-p)\delta\alpha_2$$

(v) At all histories at which i is the recipient and is offered $1-\beta_{-i} > 1-\alpha_{-i}$. Same as before this one-shot deviation is redundant.

(vi) At all histories at which 1 is the recipient and is offered $1-\beta_2 < 1-\alpha_2$. The original strategy is to reject and get $p\delta\alpha_1 + (1-p)\delta(1-\alpha_2)$. Possible one-shot deviations is: 1) accept and get $1-\beta_2$. The condition for non-profitable one-shot deviations is:

$$p\delta\alpha_1 + (1-p)\delta(1-\alpha_2) \geq 1-\beta_2 \quad \forall \beta_2 > \alpha_2$$

(vii) At all histories at which 2 is the recipient and is offered $1-\beta_1 < 1-\alpha_1$. Similarly,

$$p\delta(1-\alpha_1) + (1-p)\delta\alpha_2 \geq 1-\beta_1 \quad \forall \beta_1 > \alpha_1$$

Taking together, we solve the system of equations:

$$\begin{cases} p\delta\alpha_1 + (1-p)\delta(1-\alpha_2) = 1-\alpha_2 \\ p\delta(1-\alpha_1) + (1-p)\delta\alpha_2 = 1-\alpha_1 \end{cases}$$

and obtain

$$\alpha_1 = 1-\delta+p\delta \quad \alpha_2 = 1-p\delta$$

(b): This subquestion can be solved in the same way as the previous one, but I present a slightly more compact method with the help of the proposition above.

In this game,

$$z_1 = p_1\delta\alpha_1 + (1-p_1)\delta(1-\alpha_2)$$
$$z_2 = p_2\delta\alpha_2 + (1-p_2)\delta(1-\alpha_1)$$

and by the structure of the game, $w_i = z_i$. Using the if and only if condition in the proposition:

$$\begin{cases} p_1\delta\alpha_1 + (1-p_1)\delta(1-\alpha_2) = 1-\alpha_2 \\ p_2\delta\alpha_2 + (1-p_2)\delta(1-\alpha_1) = 1-\alpha_1 \end{cases}$$

After a ton of algebra, we obtain:

$$\alpha_1 = \frac{1-\delta+p_1\delta}{(p_1+p_2)\delta-\delta+1}$$
$$\alpha_2 = \frac{1-\delta+p_2\delta}{(p_1+p_2)\delta-\delta+1}$$

Check that $\alpha_i \geq w_i = z_i$ also holds. We are reassured that in the special case where $p_1 = 1-p_2 = p$ (both players have the same belief), it gives the same answer as in part (a).

□

3 Bargaining and the uniqueness

Question 2: Show that the equilibrium above is the unique SPE.

Proof. Instead of strategies, we consider the possible SPEs in terms of the equilibrium payoffs.

Let m_i and M_i be the infimum and supremum payoffs obtained by i in any SPE **as a proposer**. We can argue that:

$$m_i \geq 1 - \delta_j M_j \tag{1}$$

for $i = A, B$. Since $\delta_j M_j$ is the highest amount i should offer j (and to which j must accept). Similarly, we can argue that:

$$M_j \leq \max \left\{ \begin{array}{c} 1 - \delta_i m_i, \\ \delta_j(\delta_j M_j) \end{array} \right\}$$

Here, $\delta_i m_i$ is the lowest offer i could accept today, so $1 - \delta_i m_i$ is the highest possible payoff when j is the proposer **and** i accepts his proposal. On the other hand, if j makes an unacceptable offer, the max amount she can be offered tomorrow is $\delta_j M_j$. So j's discounted payoff today is no more than $\delta_j(\delta_j M_j)$.

Note that it must be:

$$\max \left\{ \begin{array}{c} 1 - \delta_i m_i, \\ \delta_j(\delta_j M_j) \end{array} \right\} = 1 - \delta_i m_i$$

Otherwise, we would have

$$M_j \leq \delta_j^2 M_j$$

which is only true if $M_j \leq 0$. However, if that's the case, we must have $1 - \delta m_i > \delta_j^2 M_j$, since $\delta, m_i < 1$ and $\delta_j^2 M_j < 0$, a contradiction. We conclude that:

$$M_j \leq 1 - \delta_i m_i \tag{2}$$

Lets put together (1) and (2) to obtain:

$$\begin{aligned} M_j &\leq 1 - \delta_i m_i \\ &\leq 1 - \delta_i(1 - \delta_j M_j) \\ &\leq 1 - \delta_i + \delta_i \delta_j M_j \\ &\Leftrightarrow M_j \leq \frac{1 - \delta_i}{1 - \delta_i \delta_j} \end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
m_j &\geq 1 - \delta_i M_i \\
&\geq 1 - \delta_i(1 - \delta_j m_j) \\
&\geq 1 - \delta_i + \delta_i \delta_j m_j \\
&\Leftrightarrow m_j \geq \frac{1 - \delta_i}{1 - \delta_i \delta_j}
\end{aligned}$$

So

$$v_j = m_j = M_j = \frac{1 - \delta_i}{1 - \delta_i \delta_j}$$

This shows that the equilibrium payoffs are uniquely defined. This implies that the strategies must also be uniquely defined as

$$\begin{aligned}
\alpha_i = v_i &= \frac{1 - \delta_j}{1 - \delta_i \delta_j} \\
1 - \alpha_i = \delta_j v_j &= \frac{\delta_j (1 - \delta_i)}{1 - \delta_i \delta_j}
\end{aligned}$$

□