1 Extensive game

Definition 1.1. An **extensive game with perfect information** $\Gamma = \langle N, H, P, (u_i) \rangle$ consists of

(i) (Finite) set of players, $N = \{1, \dots, n\}$

(ii) A set H of histories with the following properties:

  (a) An initial history $\emptyset \in H$

  (b) If $(a^k)_{k=1,\dots,K} \in H$, then $(a^k)_{k=1,\dots,L} \in H$ for all $L < K$.

  (c) If an infinite sequence $(a^k)_{k=1}^{\infty}$ satisfies $(a^k)_{k=1}^{L} \in H$ for every positive integer L, then $(a^k)_{k=1}^{\infty} \in H$.

(iii) A function $P : H \setminus Z \to N$ that assigns to each nonterminal history a member of N.

(iv) Preferences over terminal histories for all i, $u_i : Z \to \mathbb{R}$

Definition 1.2. Each member of H is a **history**; each component of a history is an **action** taken by a player. A history $(a^k)_{k=1}^{K} \in H$ is called **terminal** if it is infinite or if there is no a^{K+1} such that $(a^k)_{k=1}^{K+1} \in H$. The set of terminal histories is denoted Z.

Definition 1.3. A strategy of a player i in an extensive game with perfect information is a function

$$s_i(h) \to A(h)$$

for any $h \in H \setminus Z$ such that $P(h) = i$.

Remark. A strategy specifies an action for *each* (non-terminal) history in which a player is asked to choose an action, even for histories that, if the strategy is followed, are never reached.

Definition 1.4. Denote a strategy profile $s = (s_1, \dots, s_n)$. For each strategy profile an outcome $O(s)$ is the terminal history associated with the strategy profile.

Definition 1.5. A strategy profile, $s = (s_1, \ldots, s_n)$ is a **Nash equilibrium** if for all players i and all deviations $\hat{s}_i$,

$$u_i(s_i, s_{-i}) \geq u_i(\hat{s}_i, s_{-i})$$

where $u_i(s) = u_i(O(s))$.

Definition 1.6. The **subgame** of the extensive game with perfect information $\Gamma = \langle N, H, P, (u_i) \rangle$ that follows the history h is the extensive game $\Gamma(h) = \langle N, H|_h, P|_h, (u_i)|_h \rangle$, where $H|_h, P|_h, (u_i)|_h$ are consistent with the original game starting at history h.

Definition 1.7. A strategy profile, s is a **subgame perfect equilibrium** in Γ if for any history h the strategy profile $s|_h$ is a Nash equilibrium of the subgame $\Gamma(h)$.

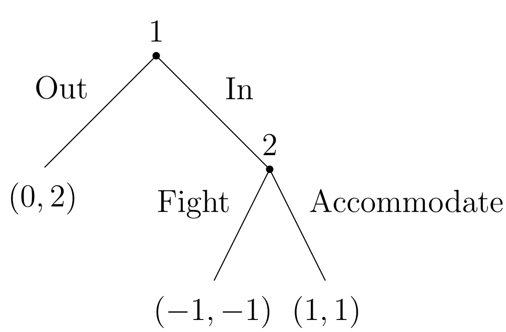
Definition 1.8. For fixed s_i and history h, a **one-stage deviation** is a strategy $\hat{s}_i$ in the subgame $\Gamma(h)$ that differs from $s_i|_h$ only in the action it prescribes after the initial history of $\Gamma(h)$.

Theorem 1.1 (One-stage deviation principle). *In a finite-horizon extensive game or infinite horizon games continuous at infinity, a strategy profile s is an SPE if and only if for all players i, all histories $h \in H$, and one-stage deviations $\hat{s}_i$,*

$$u_i(s_i|_h, s_{-i}|_h) \geq u_i(\hat{s}_i, s_{-i}|_h)$$

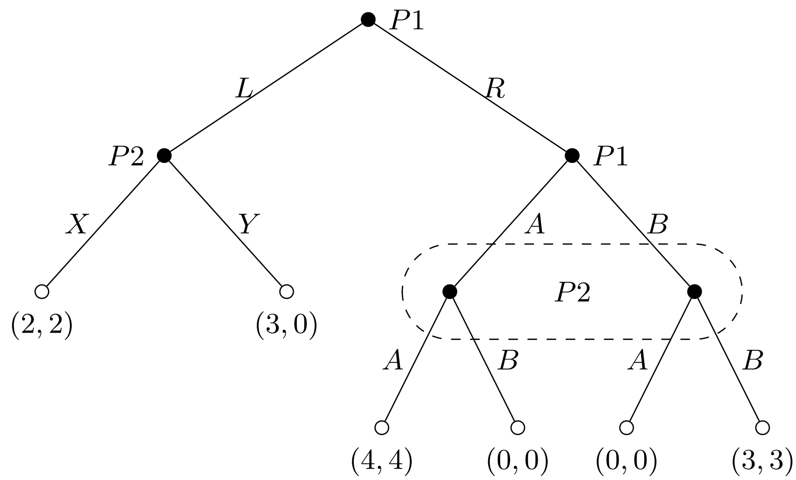
Theorem 1.2 (Kuhn's). *SPE for finite extensive games can be found by Backward induction.*

Example 1.3 (Entry game).



2 Microeconomic Theory Qualification Exam, 2018 Retake: Question III

Part III (20 Points)
Consider the following dynamic game in extensive form:



(a) (3 points) List all pure strategies that each player has.

(b) (3 points) How many subgames are there? Please describe them.

(c) (9 points) Find all (pure or mixed) subgame perfect equilibria.

(d) (5 points) Find a Nash equilibrium that is not subgame perfect.

3 Ultimatum Game

Consider the following two stage game, about how two players split a pile of 100 gold coins. The action (strategy) set of player 1 is given by $S_1 = \{0, \cdots, 100\}$, with choice i meaning that player 1 proposes to keep i of the gold coins. Player 2 learns the choice of player one, and then takes one of two actions in response: 1 (accept) or 0 (reject). If player two plays accept, the payoff vector is $(i, 100 - i)$. If player two plays reject, the payoff vector is $(0, 0)$.

(a) Describe the extensive form version of the game using a game tree.

(b) Describe the normal form of the game. It suffices to specify the strategy spaces and payoff functions. (Hint: Player 2 has 2^{101} pure strategies.)

(c) Identify a Nash equilibrium of the normal form game with payoff vector $(50, 50)$.

(d) Identify the subgame perfect equilibria of the extensive form game. (Hint: There are two of them.)

(e) Do the subgame perfect equilibria change if player 1's strategy space is now continuous, i.e., $S_1 = [0, 100]$?