ECON 6110, Microeconomic Theory III	Section 3
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1 Best response, strict domination, and rationalizability

Definition 1.1. A belief of player i about the other players actions is a probability measure over the set A_{-i}, which we denote as $\Delta(A_{-i})$.

Remark. Note the difference with $\Pi_{j\neq i}\Delta(A_j)$!

Definition 1.2. An action a_i of player i in a strategic game is a **never-best response** if for all beliefs μ_i there exists $\alpha_i \in \Delta(A_i)$ such that

$$u_i(\alpha_i, \mu_i) \geq u_i(a_i, \mu_i)$$

This means that for every belief of player i there is *some* action that is better for player i than a_i.

Definition 1.3. The action a_i of player i in the strategic game is **strictly dominated** if there is *one* mixed action α_i of player i such that for all $a_{-i} \in A_{-i}$,

$$U_i(\alpha_i, a_{-i}) > U_i(a_i, a_{-i})$$

Lemma 1.1 (Osborne & Rubinstein Lemma 60.1)**.**
Never-best response $\iff$ *strictly dominated*
Sometimes a best response $\iff$ *not strictly dominated*
(One direction is true only if we allow correlated beliefs.)

Remark. One direction is not so obvious!

Definition 1.4. An action $a_i \in A_i$ is **rationalizable** if there exists $(Z_j)_{j\in N}$ and $Z_j \subseteq A_j$ $\forall j$ such that:

- $a_i \in Z_i$;
- Every action $a_j \in Z_j$ is a best response (among A_j) to a belief $\mu_j^{a_j}$ of player j that is supported on Z_{-j}.

Remark. This definition is great for checking whether a given group of sets of strategies is rationalizable, but it doesn't tell us how to find them. The following proposition is thus what we work with most of the time.

Proposition 1.2. *If* $X = \Pi_{j \in N} X_j$ *survives iterated elimination of strictly dominated actions in a finite strategic game* $\langle N, (A_i), (u_i) \rangle$ *then* X_j *is the set of player* j*'s rationalizable actions for each* $j \in N$*. (This is true only if we allow correlated beliefs.)*

Remark. If we require beliefs to be independent, iterated elimination of strictly dominated strategies leaves a **bigger** set than rationalizable strategies. Reason: when you put restrictions on the belief $\mu_i(a_i)$, you can rationalize fewer things. But strictly dominated strategies are not about beliefs and thus not affected.

Corollary 1.3. *Let* α^* *be a mixed Nash equilibrium. For every player* i*, all actions in the support of* α_i^* *is rationalizable.*

Remark. By Proposition 1.2, To find the set of rationalizable strategies, we just need to perform iterated elimination of strictly dominated strategies (IESDS). By Lemma 1.1, we just need to delete the strategies that are never-best response and keep the strategies that are sometimes best response.

2 Exercise

Microeconomics Qualification Exam 2017 Regular: Question IV

Consider a first-price auction featuring two bidders competing for a single object. Bidder 1 values the object at \$1 and Bidder 2 values the object at \$2. After the bidders submit their bids simultaneously, the good is allocated to the winner who has to pay her bid, whereas the loser pays nothing. In the case of a tie, the winner is decided by a fair coin. The rules of auction specify that no bid is allowed to exceed \$5.

(a) What strategies are rationalizable?

(b) Consider a modified version of the game where bids are only allowed in increments of cents (\$.01). What strategies are rationalizable now?

Solution:

For this game:

- $A_1 = A_2 = [0, 5]$
- $v_1(a_1, a_2) = \begin{cases} 1 - a_1; & a_1 > a_2 \\ \frac{1-a_1}{2}; & a_1 = a_2 \\ 0; & a_1 < a_2 \end{cases}$
- $v_2(a_1, a_2) = \begin{cases} 2 - a_2; & a_2 > a_1 \\ \frac{2-a_2}{2}; & a_1 = a_2 \\ 0; & a_2 < a_1 \end{cases}$

(a) The set of rationalizable strategies for both players is $[0, 5)$. First, observe that

$$v_2(5, 0) = 0 > -\frac{3}{2} = v_2(5, 5),$$

$$v_2(0, 0) = 1 > -3 = v_2(0, 5),$$

and

$$v_2(a_1, 0) = 0 > -3 = v_2(a_1, 5)$$

for all $a_1 \in (0,5)$. Thus, $a_2 = 0$ strictly dominates $a_2 = 5$ for Player 2. A symmetric argument shows that $a_1 = 5$ is strictly dominated by $a_1 = 0$ for player 1. Once these strategies are deleted for both players, the deletion algorithm stops, as $[0, a_{-i}) = BR_i(a_{-i})$ for all $a_{-i} > 2$.

(b) In the modified game, the set of rationalizable strategies for Players 1 and 2 are

$$\{0, .01, .02, ..., 1.98\}$$

and

$$\{.01, .02, ..., 1.99\},$$

respectively:

To begin the iterated deletion algorithm, notice that both players can eliminate $a_i = 5$ for the same reason as in part (a). However, once both players eliminate $a_i = 5$, the strategy $a_i = 4.99$ becomes strictly dominated by $a_i = 0$, and can thus be deleted in the second round. The strategy $a_i = 4.98$ can similarly be deleted in the third round, and so and so forth until the remaining strategies for both players are $\tilde{A}_i = \{0, .01, ..., 2\}$. Once we reach this stage of the algorithm, Player 2 can no longer delete any further strategies as $\tilde{A}_2 = BR_2(2)$. However, Player 1 can delete $a_1 = 2$ as it is strictly dominated by $a_1 = 0$:

$$\begin{aligned} v_1(2,2) &= -1/2 < 0 = v_1(0,2) \\ v_1(2,a_2) &= -1 < 0 = v_1(0,a_2); \quad 0 < a_2 < 2 \\ v_1(2,0) &= -1 < 1/2 = v_1(0,0). \end{aligned}$$

Once $a_1 = 2$ is deleted, $a_2 = 1.99$ strictly dominates $a_2 = 2$, as $a_2 = 1.99$ earns a positive payoff against all of 1's remaining strategies whereas $a_2 = 2$ necessarily earns zero. Thus, $a_2 = 2$ is deleted, which then causes $a_1 = 0$ to strictly dominate $a_1 = 1.99$. However, once $a_1 = 1.99$ is deleted, Player 2 cannot delete any *positive* strategies. We have that $1.99 \in BR_2(1.98)$ on account of the fact that

$$\begin{aligned} v_2(1.98, 1.99) &= .01 = .02/2 = v_2(1.98, 1.98) \\ v_2(1.98, a_2) &= 0 \quad \forall a_2 < 1.98. \end{aligned}$$

Furthermore, for all $0 < a_2 \leq 1.98$, $a_2 \in BR_2(a_2 - .01)$. However, $a_2 = 0$ can be deleted. There are two ways to show it. One, we can show that $a_2 = 0$ is a never-best-response, specifically,

$$\begin{cases} 0.01 \succ 0 & \text{if } a_1 = 0 \\ 1.99 \succ 0 & \text{if } 0 < a_1 \leq 1.98 \end{cases}$$

Alternatively (by Lemma 1.1), we can show it is strictly dominated by the mixed strategy $\alpha_2' = [1/1000](a_2 = 1.99) + [999/1000](a_2 = 0.01)$:

$$v_2(0, \alpha_2') > 1.99 * (999/1000) > 1 = v_2(0, 0).$$
$$v_2(a_1, \alpha_2') > 0 = v_2(a_1, 0) \quad \forall a_1 \neq 0.$$

In fact, any weight on $a_2 = 0.01$ that's larger than 0.51 would work.

After Player 2 deletes $a_2 = 0$, Player 1 cannot delete any further strategies as

$$\{0, .01, .02, ..., 1.98\} = BR_1(1.99).$$

As such, Player 2 cannot delete any further strategies, so the algorithm ends.