ECON 6110, Microeconomic Theory III	Section 2
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1 Correlated equilibrium

Definition 1.1. A correlated equilibrium of a strategic game $\langle N, (A_i), (u_i)\rangle$ consists of:

- a finite probability space (Ω, π) (Ω is a set of states and π is a probability measure on Ω)
- for each player $i \in N$ a partition $\mathcal{P}_i$ of Ω (player i's information partition)
- for each player $i \in N$ a strategy $\sigma_i : \Omega \to A_i$ with $\sigma_i(\omega) = \sigma_i(\omega')$ whenever $\omega, \omega' \in P_i$ for some $P_i \in \mathcal{P}_i$ (equivalently, $\sigma_i : \mathcal{P}_i \to A_i$)

such that for every $i \in N$ and every alternative strategy $\tau_i : \Omega \to A$ of player i that satisfies $\tau_i(\omega) = \tau_i(\omega')$ whenever $\omega, \omega' \in P_i$, we have

$$\sum_{\omega\in\Omega} \pi(\omega) u_i(\sigma_i(\omega), \sigma_{-i}(\omega)) \geq \sum_{\omega\in\Omega} \pi(\omega) u_i(\tau_i(\omega), \sigma_{-i}(\omega)) \tag{1}$$

or

$$\sum_{\omega\in\Omega} \Pr(\omega \mid P_i) u_i(\sigma_i(\omega), \sigma_{-i}(\omega)) \geq \sum_{\omega\in\Omega} \Pr(\omega \mid P_i) u_i(\tau_i(\omega), \sigma_{-i}(\omega)) \quad \forall P_i \in \mathcal{P}_i \tag{2}$$

Proof. (2) $\implies$ (1): Law of total expectations.

(1) $\implies$ (2): Proof by contradiction. $\tau_i(\omega) = \sigma_i(\omega), \forall \omega \notin P_i$ and $\tau_i(\omega'), \forall \omega' \in P_i$. $\square$

Example:

- $N = 2$
- $A_1 = \{U, D\}$
- $A_2 = \{L, R\}$
- $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$
- $\pi = \{\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{1}{3}\}$
- $\mathcal{P}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$

- $\mathcal{P}_2 = \{\{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}\}$
- $\sigma_1(\omega_1) = \sigma_1(\omega_2) = U$
- $\sigma_1(\omega_3) = \sigma_1(\omega_4) = D$
- $\sigma_2(\omega_1) = \sigma_2(\omega_3) = L$
- $\sigma_2(\omega_2) = \sigma_2(\omega_4) = R$

is a correlated equilibrium if for every $i \in N$ and every alternative strategy $\tau_i : \Omega \to A$ of player i that satisfies $\tau_i(\omega) = \tau_i(\omega')$ whenever $\omega, \omega' \in P_i$, we have:

$$\sum_{\omega \in \Omega} \Pr(\omega \mid P_i) u_i(\sigma_i(\omega), \sigma_{-i}(\omega)) \geq \sum_{\omega \in \Omega} \Pr(\omega \mid P_i) u_i(\tau_i(\omega), \sigma_{-i}(\omega)) \quad \forall P_i \in \mathcal{P}_i$$

Suppose player 1 receives signal U. She is in the information partition P_{11}. The probability of state conditional on P_{11} is half ω_1 and half ω_2. In these two states, the opponent will play L and R, respectively. Substituting to the inequality:

$$\frac{1}{2} u_1(U, L) + \frac{1}{2} u_1(U, R) \geq \frac{1}{2} u_1(D, L) + \frac{1}{2} u_1(D, R)$$

Verify this inequality for each player and each information partition of the players would give a correlated equilibrium.

Proposition 1.1. *WLOG can assume* $\Omega = A = \Pi_i A_i$*,* $\mathcal{P}_i$ *consists of sets of the type* $\{a \in A : a_i = b_i\}$ *for some action* $b_i \in A_i$*, and* $\sigma_i(a) = a_i$*.*

Remark. The initial π would give the final distribution of outcomes if the construction is successful.

Remark. The probability space and the partitions are endogenous, part of the equilibrium definition.

Remark. Correlated equilibria can be found with linear programming. When all players in the game are *no-SWAP regret* algorithms, the empirical distribution of the play converges to a correlated equilibrium.

2 Exercise

In the following game, compute all the Nash equilibria, and find a correlated equilibrium that is not in the convex hull of the Nash equilibria.

	L	R
U	$8, 8$	$4, 9$
D	$9, 4$	$1, 1$

Answer. The Nash equilibria are (D, L), (U, R), and (3/4U + 1/4D, 3/4L + 1/4R). A correlated equilibrium is 1/3(U, L) + 1/3(U, R) + 1/3(D, L).

Since game is symmetric, check only player 1's incentive compatibility constraint:

Given being recommended U, the state is equally likely to be (U, L) or (U, R), so player 1's payoff of obeying and doing U is higher than that of deviating:

$$\frac{1}{2} \cdot 8 + \frac{1}{2} \cdot 4 \geq \frac{1}{2} \cdot 9 + \frac{1}{2} \cdot 1$$

Given being recommended D, the state can only be (D, L), and (D, L) is Nash so there is no incentive of deviating.

More generally, we can find the set of correlated equilibria by assuming a general π and solve for the inequalities that characterize π.

	L	R
U	p_1	p_2
D	p_3	$1 - p_1 - p_2 - p_3$

We can write the incentive compatibility constraint of the first player: when being recommended U, they play U rather than deviating to D.

$$8\frac{p_1}{p_1 + p_2} + 4\frac{p_2}{p_1 + p_2} \geq 9\frac{p_1}{p_1 + p_2} + 1\frac{p_2}{p_1 + p_2}$$

Or simply:

$$8p_1 + 4p_2 \geq 9p_1 + 1p_2$$

Similarly, write out the other ICs that set of CEs must satisfy:

$$\begin{cases} 8p_1 + 4p_2 \geq 9p_1 + 1p_2 \\ 9p_3 + 1(1 - p_1 - p_2 - p_3) \geq 8p_3 + 4(1 - p_1 - p_2 - p_3) \\ 8p_1 + 4p_3 \geq 9p_1 + 1p_3 \\ 9p_2 + 1(1 - p_1 - p_2 - p_3) \geq 8p_2 + 4(1 - p_1 - p_2 - p_3) \end{cases}$$

Note that in this case player 1 and 2 are not symmetric anymore because the π is not necessarily symmetric.