ECON 6110, Microeconomic Theory III	Section 1
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1 A few results about mixed strategies

Definition 1.1.

- $N = \{1, \ldots, n\}$: set of players.
- A_i: set of actions of player i.
- $A = \Pi_{i=1}^n A_i$: set of profiles of actions. $\times$ is the Cartesian product of sets. $A \times B$ is the set of all ordered pairs of (a, b) where $a \in A, b \in B$.
- $a = (a_1, \ldots, a_n) \in A$.

Definition 1.2.

- $\alpha_i : A_i \to [0, 1]$ such that $\sum_{a_i} \alpha_i(a_i) = 1$: probability distribution over i's actions. We also call α_i a mixed action.
- $\Delta(A_i)$: set of probability distribution over i's actions. $\alpha_i \in \Delta(A_i)$.
- $\Pi_{i=1}^n \Delta(A_i)$: set of profiles of mixed actions.
- $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Pi_{i=1}^n \Delta(A_i)$.

Remark. $\alpha_i(U) = \alpha_i(D) = \frac{1}{2}$, can also write as a shorthand $\alpha_i = \frac{1}{2}U + \frac{1}{2}D$.

Remark. $\Pi_{i=1}^n \Delta(A_i) \neq \Delta(\Pi_{i=1}^n A_i) = \Delta(A)$.

Remark. In the definition of Nash equilibrium, the domain of the best response correspondence is $\Pi_{j \neq i} \Delta(A_j)$. In correlated equilibrium and rationalizable actions, the domain of the belief is $\Delta(A_{-i}) = \Delta(\Pi_{j \neq i} A_j)$.

Definition 1.3 (Mixed Nash equilibrium)**.** A mixed Nash equilibrium of a game $\langle N, (A_i), (u_i) \rangle$ is a profile $\alpha^* = (\alpha_1^*, \ldots, \alpha_n^*)$ of mixed actions such that for every $i \in N$:

$$U_i(\alpha_i^*, \alpha_{-i}^*) \geq U_i(\alpha_i, \alpha_{-i}^*)$$

for all $\alpha_i \in \Delta(A_i)$.

Definition 1.4 (Two ways to decompose the utility of mixed actions)**.**

$$\begin{aligned} U_i(\alpha) &= \sum_{a_1, \ldots, a_n} u_i(a_1, \ldots, a_n) \cdot \alpha_1(a_1) \cdots \alpha_n(a_n) \\ &= \sum_{a_i} U_i(a_i, \alpha_{-i}) \cdot \alpha_i(a_i) \end{aligned}$$

Definition 1.5. $supp(\alpha_i) = \{a_i \in A_i : \alpha_i(a_i) > 0\}$.

Proposition 1.1 (Randomize out of indifference)**.** *If $\alpha_i^* = (\alpha_i^*, \alpha_{-i}^*)$ is a Nash equilibrium, then*

$$U_i(a_i^*, \alpha_{-i}^*) = U_i(\alpha_i^*, \alpha_{-i}^*)$$

for all $a_i^* \in supp(\alpha_i^*)$.

Proposition 1.2. *For every i and $\alpha = (\alpha_i, \alpha_{-i})$, the followings are equivalent:*

1. *For all $\alpha_i' \in \Delta(A_i)$,*
$$U_i(\alpha_i, \alpha_{-i}) \geq U_i(\alpha_i', \alpha_{-i})$$
2. *For all $a_i \in supp(\alpha_i)$ and $\alpha_i' \in \Delta(A_i)$,*
$$U_i(a_i, \alpha_{-i}) \geq U_i(\alpha_i', \alpha_{-i})$$
3. *For all $a_i \in supp(\alpha_i)$, $a_i' \in A_i$, and $\alpha_i' \in \Delta(A_i)$,*
$$U_i(a_i, \alpha_{-i}) \geq U_i(a_i', \alpha_{-i})$$

2 Exercise

Find all mixed strategy Nash equilibria of the following game.

	L	R
U	$8, 3$	$3, 1$
M	$7, 5$	$4, 4$
D	$3, 3$	$7, 5$